Lecture(s) Feb 8.

The Levi form.

Let $M \subseteq \mathbb{C}^{n+1}$ be a real (smooth) hypersurface, and $p \in M$. Let ρ be defining fcn near p, i.e. M is given by $\rho = 0$ and $d\rho \neq 0$ on M.

<u>Def 1.</u> <u>The Levi form</u> of M w.r.t. ρ at p is the Hermitian form

$$\mathcal{L}_p^\rho : T_p^{1,0}M \times T_p^{1,0}M \to \mathbb{C}$$

given by

$$\mathcal{L}_p^\rho(X_p, Y_p) = \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p) X_p^i \overline{Y_p^j},$$

where $X_p = \sum_i X_p^i \frac{\partial}{\partial z_i}$, $Y_p = \sum_j Y_p^j \frac{\partial}{\partial z_j}$.

Prop 1. Let $X_p, Y_p \in T_p^{1,0}M$, and let X, Y be sections (v.f.) of $T^{1,0}M$ extending X_p, Y_p near p. Then

$$\mathcal{L}_p^{\rho}(X_p, Y_p) = \sqrt{-1}\, \partial\rho([X, \bar{Y}])\big|_p$$

Pf. Set $\theta = \sqrt{-1}\,\partial\rho$. Then we note that

$$d\theta = \sqrt{-1}\, d\partial\rho = \sqrt{-1}\,\bar{\partial}\partial\rho = \sqrt{-1}\sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}\, dz_i \wedge d\bar{z}_j$$

Thus, in diff. form notation, we can write

$$\mathcal{L}_p^{\rho}(X_p, Y_p) = -d\theta(X, \bar{Y})\big|_p$$

<u>Cartan's "Magic" formula:</u>

$$d\theta(X, \bar{Y}) = X\theta(\bar{Y}) - \bar{Y}\theta(X) - \theta([X, \bar{Y}]).$$

Now, $\theta(X) = \sum_i \frac{\partial \rho}{\partial z_i} X^i = 0$ since X is (*) a section of $T^{1,0}M$, and similarly $\theta(\bar{Y}) = 0$.

The conclusion follows. $\boxtimes$

(*) see end of notes.

The way the Levi form is defined it appears to depend on the coordinates z. Let's look into this:

Let $w = H(z)$, where H is a biholom. (i.e. $H = (H_1, \dots, H_N)$ with $N = n+1$ a hol. invertible map near p; $p' = H(p)$)

Chain rule:

$$\frac{\partial \rho}{\partial z_i} = \sum_l \frac{\partial H^l}{\partial z_i} \frac{\partial \rho}{\partial w^l}$$

Again:

$$\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} = \sum_{l,k} \frac{\partial H^l}{\partial z_i} \overline{\frac{\partial H^k}{\partial z_j}} \frac{\partial^2 \rho}{\partial w_l \partial \bar{w}_k}$$

since H is holom. $\Rightarrow \frac{\partial H^l}{\partial z_i}$ is holom.

$$\Rightarrow \frac{\partial}{\partial \bar{z}_j}\left(\frac{\partial H^l}{\partial z_i}\right) = 0$$

Thus, w/ $X_p, Y_p \in T_p^{1,0}M \Rightarrow$

$$\sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} X_p^i \overline{Y_p^j} = \sum_{k,l} \sum_{i,j} \frac{\partial^2 \rho}{\partial w_k \partial \bar{w}_l} \frac{\partial H^k}{\partial z_i} \overline{\frac{\partial H^l}{\partial z_j}} X_p^i \overline{Y_p^j}$$

Now
$$\sum_k \sum_i \frac{\partial H^k}{\partial z_i} X_p^i \frac{\partial}{\partial w_k} = H_*(X_p),$$

or in other words, simply X_p expressed in the coordinates w, if

$$X_p = \sum_{i=1}^n \tilde{X}_p^i \frac{\partial}{\partial w_i}, \text{ then}$$

$$\tilde{X}_p^k = \sum_i \frac{\partial H^k}{\partial z_i} X_p^i$$

With similar notation for Y, we conclude

$$\sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} X_p^i \overline{Y_p^j} = \sum_{k,l} \frac{\partial^2 \tilde{\rho}}{\partial w_k \partial \bar{w}_l} \tilde{X}_p^k \overline{\tilde{Y}_p^l}$$

We have proved

<u>Prop 2</u> The Levi form of M w.r.t ρ is independent of the local coordinates chosen.

The Levi form of M does depend on ρ.

Let $\tilde{\rho}$ be another defining fcn.

Then $\exists$ smooth fcn $\overset{\circ}{a}$ s.t. $\tilde{\rho} = a\rho$.

We compute:

$$\frac{\partial \tilde{\rho}}{\partial z_i} = a \frac{\partial \rho}{\partial z_i} + \rho \frac{\partial a}{\partial z_i} \quad \text{and}$$

$$\frac{\partial^2 \tilde{\rho}}{\partial z_i \partial \bar{z}_j} = a \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} + \frac{\partial a}{\partial \bar{z}_j} \frac{\partial \rho}{\partial z_i} + \frac{\partial a}{\partial z_i} \frac{\partial \rho}{\partial \bar{z}_j}$$

$$+ \rho \frac{\partial^2 a}{\partial z_i \partial \bar{z}_j}$$

We note that all terms in RHS vanish when we choose $p \in M$ and $X_p, Y_p \in T_p^{1,0}M$. The last term vanishes since $\rho = 0$ at p. The 2^{nd} and 3^{rd} terms vanish b/c $X_p, Y_p \in T_p^{1,0}M$ as in the previous proof. Thus:

<u>Prop 3.</u> If $\tilde{\rho} = a\rho$ is another defining function, then

$$\mathcal{L}_p^{\tilde{\rho}} = a\mathcal{L}_p^{\rho}.$$

So what is invariant on M?

<u>Def 2.</u> Choose an orientation of M and let <u>the signature</u> $(a,b,r)_p$ at p be # of

$+,-,0$ eigenvalues of the Levi form at p wrt any defining fcn consistent with the orientation.

<u>Prop 4.</u> If M is an oriented real hypersurface, then the signature $(a,b,r)_p$ is invariant.

<u>Pf.</u> Follows from Prop's 3,4 since any two defining functions consistent w/ the orientation are related by $\tilde{\rho} = a\rho$ with $a > 0$. ∎

<u>Note:</u> Here, for a hypersurface, I think of a choice of orientation as a choice of unit normal vector N (w.r.t Euclidean metric on $\mathbb{C}^n \cong \mathbb{R}^{2n}$), and ρ is consistent if $\nabla\rho \cdot N > 0$.

Rem. Without orientation we cannot distinguish between a, b in $(a, b, r)_p$, but r is invariant and $|a-b|$ is invariant

Def 30 ① A hypersurface $M \subseteq \mathbb{C}^{n+1}$ is Levi nondegenerate at $p \in M$ if $r = 0$, (Levi form is nondegenerate).

② M is strictly pseudoconvex at $p \in M$ if $r = 0$ and either a or b is 0 (Levi form is positive or negative definite, depending on orientation).

Rem. Since the CR dim of $M \subseteq \mathbb{C}^{n+1}$ is n, we always have $a+b+r=n$. Thus, strict ψcvxity can be expressed invariantly wrt $(a,b,r)_p$ by $|a-b|=n$.

(*) Clarification of $\theta(\bar{Y})=0$ in pf of Prop 1. Although $d\rho \neq 0$, $d\rho(W)=0$ for all $W \in \mathbb{C}\otimes TM$ since $\rho=0$ on M. Hence, since $d=\partial+\bar\partial$, $\partial\rho(W)=-\bar\partial\rho(W) \Rightarrow$

$$\sqrt{-1}\,\partial\rho(\bar{Y}) = -\sqrt{-1}\,\bar\partial\rho(\bar{Y}) = \overline{\sqrt{-1}\,\partial\rho(Y)}=0.$$